Transition caused by multiplicative noises for finite globally coupled oscillators

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We study an ensemble of N globally coupled oscillators driven simultaneously by additive and multiplicative noises. By the numerical simulation, we find that there is a transition caused by multiplicative noises, which is different from the one proposed by Pikovsky *et al.* [Z. Phys. B **95**, 541 (1994)]. The difference between them is that the former is caused by the change of the intensity of the multiplicative noises, while the latter is caused by the change of the coupling constant. The two transitions both disappear when the number of oscillators tends to infinity. For the former transition, with the increase of the multiplicative noise strength the order parameter will first increase; then when it approaches a crest value, it will quickly attenuate to zero. [S1063-651X(98)06208-4]

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I. INTRODUCTION

There has been an increasing interest in the influence of noises on transitions and bifurcations [1–17]. A large number of coupled models have been proposed [2–17]. These models are divided into two types. One type is the locally coupled models in which each oscillator is influenced only by its neighbors [2–9]. The other type is the globally coupled models in which the interaction does not depend on the distance between elements [10–17].

In Ref. [10] Pikovsky *et al*. investigated numerically the collective behavior of overdamped nonlinear noise-driven oscillators globally coupled via a mean field. They found that when a coupling constant is increased, a transition in the dynamics of the mean field is observed and this transition disappears when the number of oscillators tends to infinity. However, Pikovsky *et al.* studied only the additive noise case. In the paper we will introduce in Sec. II a model with finite globally coupled oscillators driven simultaneously by additive and multiplicative noises. In Sec. III the transition for this model will be studied. A transition caused by the change of the multiplicative noise strength will be found.

II. MODEL

Consider a system that has N identical oscillators. Their differential equation is

$$\frac{dx_i}{dt}=\epsilon s x_i - x_i^3 \quad (i=1,\ldots,N), \tag{1}$$

where s is the mean field defined as $s=(1/N)\Sigma_i^N x_i$ and ϵ is the coupling constant. If we consider only the internal noises induced by the internal origin, Eq. (1) becomes

$$\frac{dx_i}{dt}=\epsilon s x_i - x_i^3 + \eta_i(t), \tag{2}$$

where $\eta_i(t)$ is Gaussian white noise with zero mean and the correlation function $\langle \eta_i(t)\eta_j(t')\rangle = 2D_2\delta_{ij}\delta(t-t')$. Equation (2) is the model studied in Ref. [10]. We consider the case when the system is influenced by the internal and external noises. The differential equation of the oscillators is

$$\frac{dx_i}{dt}=\epsilon s x_i - x_i^3 + x_i\xi_i(t) + \eta_i(t), \tag{3}$$

in which $\xi_i(t)$ is Gaussian noise, white in time and space with the statistical properties $\langle \xi_i(t)\rangle = 0$, $\langle \xi_i(t)\xi_j(t')\rangle = 2D_1\delta_{ij}\delta(t-t')$, and $\langle \xi_i(t)\eta_j(t')\rangle = 0$. It is clear that Eq. (3) is defined in the sense of Stratonovich calculus.

III. TRANSITION

A. Numerical simulation

By numerical calculus we find that for Eq. (3) there is the same transition in the dynamics of the mean field when the coupling constant is increased just as that proposed by Pikovsky *et al.* in Ref. [10]. In order to avoid unnecessary repetition we shall not present figures that are basically similar to the ones in Ref. [10]. In Ref. [11] we investigated the effect of the multiplicative noises on this transition. In this paper we shall show a different phenomenon: a transition caused by the multiplicative noises.

In Fig. 1 we plot the behavior of s numerically with respect to t. The figures are obtained using the method explained in the Appendix. The number of oscillators in the ensemble is kept at 50. For convenience, we always set the intensity of the additive noises $D_2=2$ and the globally coupling constant $\epsilon=10$. The average value of the mean field (the order parameter) $Z=\langle s\rangle$ (angular brackets denote averaging over time) versus the intensity of the multiplicative noises curves is represented in Fig. 2, in which the number of oscillators is $N=50$, 100, and 150. The figures show that

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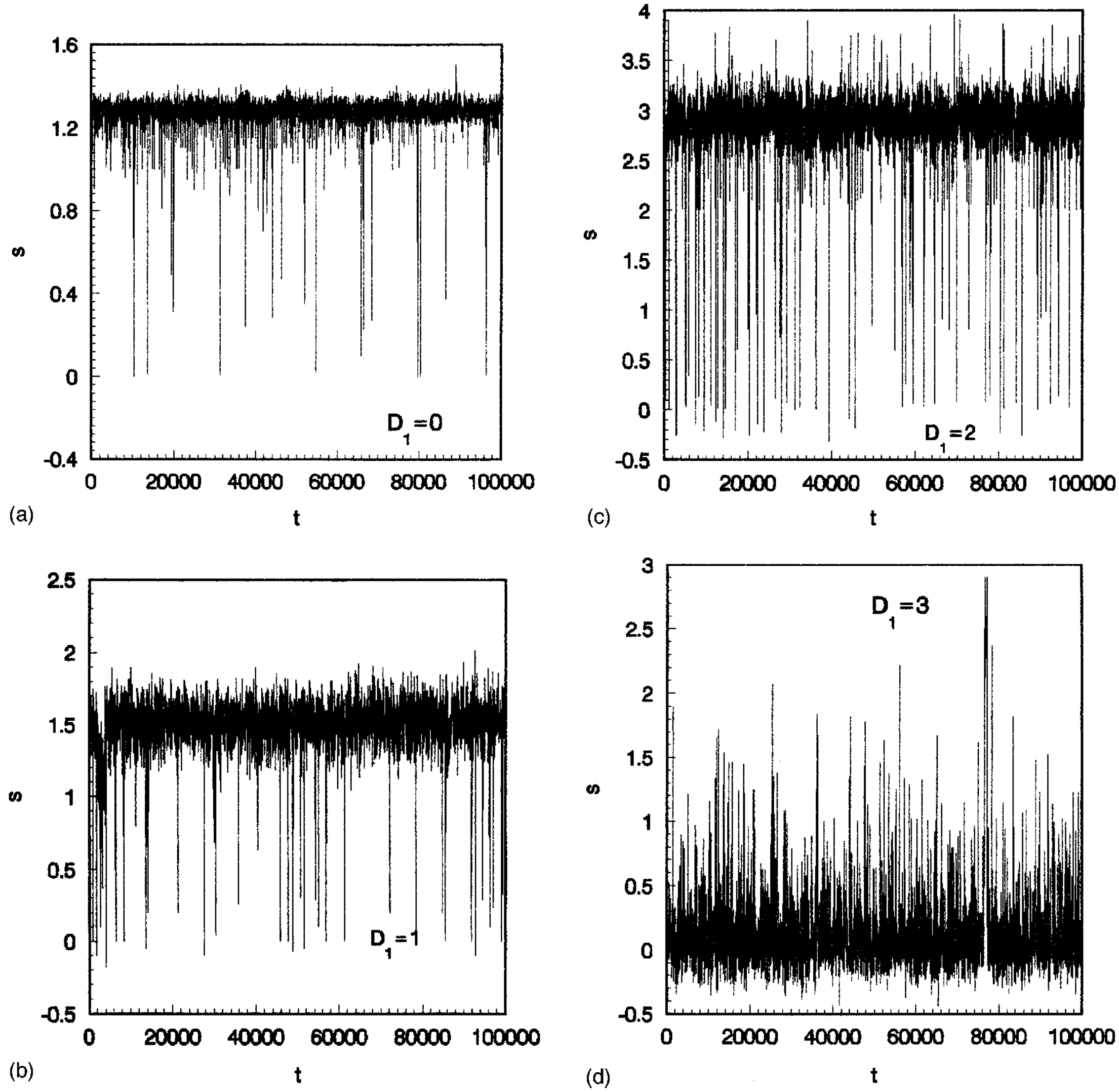

FIG. 1. Fluctuations of the mean field s. The number of oscillators in the ensemble $N=50$, the intensity of the additive noises $D_2=2$, and the coupling constant $\epsilon=10$.

there is a transition in the dynamics of the mean field with the increase of the intensity of the multiplicative noises, which is different from the one proposed by Pikovsky *et al.* [10]. The difference between them is that the former is caused by the change of the intensity of the multiplicative noises, while the latter is caused by the change of the coupling constant. For the former, with the increase of the multiplicative noise strength the order parameter will first increase; then when it approaches a crest value, it will quickly attenuate to zero (cf. Fig. 2). For the latter, with an increase of the coupling constant, the order parameter will monotonically increase from zero to a definite constant (cf. Fig. 3 in Ref. [10]). In addition, from Fig. 2 we find that with the increase of N the transition caused by the change of the intensity of the multiplicative noises becomes more and more indistinct, which is similar to the one caused by the change of the coupling constant (cf. Fig. 3 in Ref. [10]).

The above results are obtained by numerical simulation. Below we shall carry out some theoretical analysis for Eq. (3).

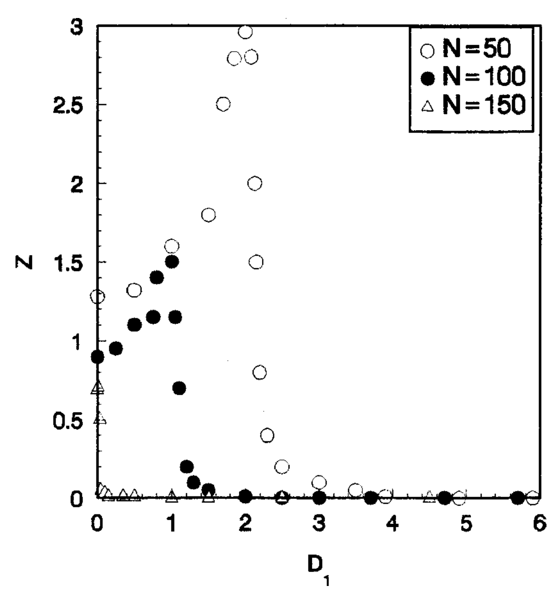

FIG. 2. Average mean field $Z=\langle s\rangle$ versus the intensity of the multiplicative noises D_1. $D_2=2$ and $\epsilon=10$.

B. Theoretical analysis

Let us first consider the case of $N\to\infty$. In this case we can drop the subscript i in the following since Eq. (3) is similar for every site i. So Eq. (3) becomes

$$\frac{dx}{dt}=\epsilon s x-x^3+x\xi(t)+\eta(t). \tag{4}$$

When $s=1$, Eq. (3) has been investigated intensively and it is an archetypical bistable model with an overdamped particle moving in a double well potential $V(x)=x^4/4-(\epsilon/2)x^2$ ($\epsilon>0$). The term $x\xi(t)$ can be understood as the motion of a Brownian particle on the background of a medium whose density distribution is directly proportional to $|x|$.

In the limit of $N\to\infty$, the self-consistent Weiss mean-field approach of Desai and Zwanzig is valid [18] and the Weiss mean field $Z=\langle x\rangle=s$. The system can be described by a nonlinear Fokker-Planck equation for the probability density $P(x,t)$ [19,20],

$$\partial_t P(x,t)=-\partial_x F(x)P(x,t)+\partial_x^2 D(x)P(x,t), \tag{5}$$

with $F(x)=(\epsilon s+D_1)x-x^3$ and $D(x)=D_2+D_1x^2$. The stationary solution of Eq. (5) under the natural boundary condition is [19,20]

$$P_{st}(x)=M^{-1}(D_2+D_1x^2)^c\exp[-x^2/2D_1], \tag{6}$$

where $c=(\epsilon s-D_1+D_2/D_1)/2D_1$ and M is the normalization constant. Obviously, $P_{st}(x)$ is symmetrical, so we can get $Z=\langle x\rangle=s=\int_{-\infty}^{\infty}xP_{st}(x)dx=0$. Thus there is no phase transition in the limit $N\to\infty$ (the thermodynamic limit). In the following, we shall consider the case when the number of oscillators is finite.

If $N=1$ and $s=x$, the Langevin equation (3) turns into

$$\frac{dx}{dt}=\epsilon x^2-x^3+x\xi(t)+\eta(t). \tag{7}$$

The Fokker-Planck equation for the probability density of the diffusion process, defined by Eq. (7), reads

$$\begin{aligned}\partial_t P(x,t)=&-\partial_x(\epsilon x^2+D_1x-x^3)P(x,t)\\&+\partial_x^2(D_2+D_1x^2)P(x,t).\end{aligned} \tag{8}$$

By analyzing Eq. (8) and its stationary solution further, some valuable information can be obtained. The extrema of the stationary probability $P_{st}(x)$ can be determined from $dP_{st}(x)/dx=0$. Now $x=0$ is always an extremum and the others are given by the solutions of the equation

$$x^2-\epsilon x+D_1=0 \quad \text{for} \quad D_1\neq 0. \tag{9}$$

When $D_1<\epsilon^2/4$, the stationary state will be bistable and $x=0$ and $x=(\epsilon+\sqrt{\epsilon^2-4D_1})/2$ [or $x=(\epsilon-\sqrt{\epsilon^2-4D_1})/2$] are the maximum points. When $D_1>\epsilon^2/4$, the stationary state for the system will be monostable and $x=0$ is the maximum point for the stationary probability density. Thus we say that if $D_1\neq 0$ there is a transition for the stationary probability with the change of D_1 (the multiplicative noise strength). It is clear that the reason why the transition for Eq. (7) happens is the action of the coupling term in the Langevin equation. If N is small, the coupling term still has this action on the transition of the system. With the increase of N, this action becomes more and more poor. When $N\to\infty$, this action will disappear.

From the above study of Eq. (3) we know that there is a critical value of D_1 when N is finite. When D_1 is larger than the critical value, the system will be in the state with a zero average mean field. When D_1 is smaller than the critical value, the system will be in the state with a nonzero average mean field. We can roughly estimate the critical value of D_1. From Fig. 2 one observes that when N tends to infinity the critical value of D_1 appears to go to zero. In the special case of $N=1$, the critical value is $D_1=\epsilon^2/4$. In the case when the number of oscillators in the ensemble is N, we can approximately set the critical value to

$$D_1=D_1^{(0)}\sim\frac{K(\epsilon)\epsilon}{N^{\alpha}}, \tag{10}$$

where $K(\epsilon)$ is a function of ϵ and α is a positive constant.

APPENDIX

From Ref. [21] we can get the numerical algorithm of Eq. (3)

$$\begin{aligned}x_i(t+\Delta t)=&x_i(t)+[\epsilon s(t)x_i(t)-x_i^3(t)]\Delta t+x_1^{(i)}(t,\Delta t)\\&+x_2^{(i)}(t,\Delta t),\end{aligned} \tag{A1}$$

where $x_1^{(i)}(t,\Delta t)=[D_1x_i(t)(\phi_1^{(i)})^2+D_1D_2\phi_1^{(i)}\phi_2^{(i)}]\Delta t$ and $x_2^{(i)}(t,\Delta t)=x_i(t)\sqrt{2D_1\Delta t}\phi_1^{(i)}+\sqrt{2D_2\Delta t}\phi_2^{(i)}$. $\phi_1^{(i)}$ and $\phi_2^{(i)}$ are two independent Gaussian random numbers of zero mean and variance equal to 1. Moreover, we know that the stochastic Runge-Kutta method [22] is more accurate than the one-step Euler method. If we consider the Runge-Kutta method, the algorithm (A1) should be changed as

$$\begin{aligned}x_i(t+\Delta t)=&x_i(t)+\frac{1}{2}[\epsilon s(t)x_i(t)-x_i^3(t)+f_i(t,\Delta t)]\Delta t\\&+x_1^{(i)}(t,\Delta t)+x_2^{(i)}(t,\Delta t),\end{aligned} \tag{A2}$$

in which

$$\begin{aligned}f_i(t,\Delta t)=&\epsilon s(t)\{x_i(t)+[\epsilon s(t)x_i(t)-x_i^3(t)]\Delta t+x_1^{(i)}(t,\Delta t)\\&+x_2^{(i)}(t,\Delta t)\}-\{x_i(t)[\epsilon s(t)x_i(t)-x_i^3(t)]\Delta t\\&+x_1^{(i)}(t,\Delta t)+x_2^{(i)}(t,\Delta t)\}^3,\end{aligned}$$

$x_1^{(i)}(t,\Delta t)$, and $x_2^{(i)}(t,\Delta t)$ are similar to those in Eq. (A1). Equation (A2) is the algorithm that we have used in this paper.

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